Friedmann–Lemaitre Cosmologies via Roulettes and Other Analytic Methods

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Abstract

In this work a series of methods are developed for understanding the Friedmann equation when it is beyond the reach of the Chebyshev theorem. First it will be demonstrated that every solution of the Friedmann equation admits a representation as a roulette such that information on the latter may be used to obtain that for the former. Next the Friedmann equation is integrated for a quadratic equation of state and for the Randall–Sundrum II universe, leading to a harvest of a rich collection

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of new interesting phenomena. Finally an analytic method is used to isolate the asymptotic behavior of the solutions of the Friedmann equation, when the equation of state is of an extended form which renders the integration impossible, and to establish a universal exponential growth law.

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Contents

1 Introduction

When a homogeneous and isotropic perfect-fluid universe hypothesis is taken, the Einstein equations of general relativity are reduced into a set of linear ordinary differential equations, governing the Hubble parameter in terms of a scale factor, known as the Friedmann equations, which are of basic importance in evolutionary cosmology. In the recent systematic works [1, 2], we explored a link between the Chebyshev theorem [3, 4] and the integrability of the Friedmann equations and obtained a wealth of new explicit solutions when the barotropic equation of state is either linear or nonlinear, in both cosmic and conformal times, and for flat and non-flat spatial sections with and without a cosmological constant. The purpose of the present paper is to present several analytic methods which may be used to obtain either exact expressions or insightful knowledge of the solutions of the Friedmann equations beyond the reach of the Chebyshev theorem. These methods may be divided into three categories: roulettes, explicit integrations, and analytic approximations.

The idea of roulettes offers opportunities for obtaining some special solutions. Recall that many textbooks mention the fact that the graph of the scale factor $a(t)$ against time t for a closed Friedmann–Lemaitre cosmology supported by a fluid whose pressure is negligible follows is cycloid, thus providing an attractive image of the wheel of time steadily rolling from what Fred Hoyle dubbed a Big Bang to a Big Crunch reminding one of Huyghens' isochronous pendulum and inviting speculations, such as those of Richard Chace Tolman about cosmologies which are cyclic or cyclic up to a rescaling. More recently such models have been called Ekpyrotic following an old tradition of the Stoics.

Few of those textbooks give details or a derivation of the roulette construction of a cycloid and none, as far as we are aware, explain whether or not it is a mathematical fluke or whether any plot of the scale factor against time may be represented as a *roulette*, that is as the locus of a point on or inside a curve which rolls without slipping along a straight line.

Evidence that this may be so is provided by what is often called a *Tolman Universe*, that is one in which the fluid is a radiation gas whose energy density ρ and Pressure P are related by $\rho = \frac{1}{3}P$. In this case, for a closed universes, $a(t)$ is a semi-circle which of course

may be obtained as a sort of limiting case by "rolling without slipping" the straight line interval given by it radius. Further evidence is provided by De-Sitter spacetime which contains just *Dark Energy*. In its closed form, the scale factor has the form of a *catenary* which is the locus of the focus of a *parabola* rolled without slipping along its *directrix*. In the light of this *lacuna* in the literature, it is here proposed to provide a general discussion, showing among other things that a roulette construction is always possible and how in principle to obtain the form of the rolling curve. In order to do so the article commences by recalling some of the classical theory of plane curves, mainly developed in the 17^{th} and 18^{th} century, although its roots go back to the Greeks. As the theory is then applied to the *Friedmann and Raychaudhuri equations* familiar in cosmology it will become apparent that the ideas have a much wider relevance. In particular they extend immediately to the motion of light, or sound rays governed by *Snell's Law* moving in a vertical stratified medium with variable refractive index, and to the familiar central orbit problems of mechanics. More surprisingly perhaps we may, as Delaunay first pointed out in the 19^{th} century, we can also use the same description to describe the shapes of axisymmetric soap films [5, 6]. In this way we obtain a framework for understanding in a unified geometric fashion a wide variety of iconic problems in physics.

Recall also that the Chebyshev theorem applies only to integrals of binomial differentials [1, 2]. In cosmology, one frequently encounters models which cannot be converted into such integrals. Hence it will be useful to present some methods which may be used to integrate the Friedmann equations whose integrations involve non-binomial differentials. Here we illustrate our methods by integrating the model when the equation of state is quadratic [7, 8, 9] and the Randall–Sundrum II universe [10, 11, 12, 13] with a non-vanishing cosmological constant for which both the energy density and its quadratic power are contributing to the right-hand side of the Friedmann equation. A quadratic equation of state introduces a larger degree of freedom for the choice of parameters realized as the coefficients of the quadratic function. In the flat-space case we show that for the solutions of cosmological interest to exist so that the scale factor evolves from zero to infinity the coefficients must satisfy a *necessary and sufficient condition* that confines the ranges of the parameters. We also show that when the quadratic term is present in the equation of state the *scale factor cannot vanish at finite time*. In other words, in this situation, it requires an infinite past duration for the scale factor to vanish. As another example that cannot be dealt with by the Chebyshev theorem, we study the Randall–Sundrum II universe when the space is flat and n-dimensional and the cosmological constant Λ is arbitrary. There are two cases of interest: (i) the equation of state is linear, $P = A\rho$, $A > -1$,and (ii) the equation of state is that of the Chaplygin fluid type, $P = A\rho - \frac{B}{\rho}$, $A > -1, B > 0$. In the case (i) the big-bang solutions for $\Lambda = 0$ and $\Lambda > 0$ are similar as in the classical situation so that the scale factor either enjoys a power-function growth law or an exponential growth law according to whether $\Lambda = 0$ or $\Lambda > 0$. When $\Lambda < 0$, however, we unveil the fact that the solution has only a *finite lifespan* when $A \geq -1 + \frac{1}{n}$, and we determine the lifespan explicitly. When $-1 < A < -1+\frac{1}{n}$, the solution is periodic, as in the classical case. In the case (ii) with $\Lambda = 0$, we find the explicit big-bang solution by integration and display its exponential growth law in terms of various physical parameters in the model. Finally, we

carry out a study of the big-bang solution for the Friedmann equation when the equation of state is of an extended Chaplygin fluid form [14, 15, 16], $P = f(\rho) - \sum_{k=1}^{m} \frac{B_k}{\rho^{\alpha_k}}$, where $f(\rho)$ is an analytic function and $B_1, \dots, B_m > 0, \alpha_1, \dots, \alpha_m \geq 0$, for which an integration by whatsoever means would be impossible in general. We shall identify an explicit range of Λ for which the scale factor grows exponentially fast and deduce a universal formula for the associated exponential growth rate. This formula covers all the explicitly known concrete cases.

The rest of this article is organized as follows. In §2 we review the Friedmann equation. In §3 we demonstrate a relation between roulettes and the Friedmann equation and show how to use this relation to find some special solutions. In §4 we consider the integration of the Friedmann equation when the equation of state is quadratic. In §5 we solve the Friedmann equation for the Randall–Sundrum II universe. In §6 we establish a universal exponential growth law for the extended Chaplygin fluid universe. In §7 we summarize our results. In Appendix (§8) we collect some relevant concepts and facts used in this article.

2 Friedmann's equations

For generality we shall consider an $(n+1)$-dimensional homogeneous and isotropic space-time with the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t) g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad i, j = 1, \dots, n, \tag{2.1}$$

where g_{ij} is the metric of an n-dimensional Riemannian manifold M of constant scalar curvature characterized by an indicator, $k = -1, 0, 1$, so that M is an n-hyperboloid, the flat space $\mathbb{R}^n$, or an n-sphere, with the respective metric

$$g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{1-kr^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{n-1}^2, \tag{2.2}$$

where $r > 0$ is the radial variable and $\mathrm{d}\Omega_{n-1}^2$ denotes the canonical metric of the unit sphere S^{n-1} in $\mathbb{R}^n$, and t is referred to as the cosmological (or cosmic) time. The Einstein equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_n T_{\mu\nu}, \tag{2.3}$$

where $G_{\mu\nu}$ is the Einstein tensor, G_n the universal gravitational constant in n dimensions, and Λ the cosmological constant, the speed of light is set to unity, and $T_{\mu\nu}$ is the energy-momentum tensor. In the case of an ideal cosmological fluid, $T_{\mu\nu}$ is given by

$$T_\mu^\nu = \mathrm{diag}\{-\rho_m, P_m, \dots, P_m\}, \tag{2.4}$$

with ρ_m and p_m the t-dependent matter energy density and pressure. Thus, assuming an ideal-fluid homogeneous and isotropic universe, the Einstein equations are reduced into the Friedmann equations

$$H^2 = \frac{16\pi G_n}{n(n-1)}\rho - \frac{k}{a^2}, \tag{2.5}$$

$$\dot{H} = -\frac{8\pi G_n}{n-1}(\rho + P) + \frac{k}{a^2}, \tag{2.6}$$

in which $H = \frac{\dot{a}}{a}$ denotes the usual Hubble 'constant' with $\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t}$, and ρ, P are the effective energy density and pressure related to ρ_m, P_m through

$$\rho = \rho_m + \frac{\Lambda}{8\pi G_n}, \quad P = P_m - \frac{\Lambda}{8\pi G_n}. \tag{2.7}$$

Besides, in view of (2.1) and (2.4) and (2.7), we see that the energy-conservation law $\nabla_\nu T^{\mu\nu} = 0$ reads

$$\dot{\rho}_m + n(\rho_m + P_m)H = 0. \tag{2.8}$$

It is readily checked that (2.5) and (2.8) imply (2.6). Hence it suffices to consider (2.5) and (2.8).

In the rest of the paper, we omit the subscript m in the energy density and pressure when there is no risk of confusion.

3 Friedmann's equations and roulettes

In [2], numerous Friedmann type equations arising in a wide variety of applied areas of physics, other than cosmology, are integrated as well in view of the Chebyshev theorem, which include the equation for light refraction in a horizontally stratified medium, equations for soap films and glaciated valleys, equation of catenary of equal strength, the elastica of Bernoulli and the capillary curves, central orbit equations, equations of capillary curves, equations for spherically symmetric lenses, etc. These equations may also be recast in the forms of roulettes as seen below.

3.1 Rolling without slipping

In what follows, extensive use has been made of [17]. Suppose a closed convex curve γ enclosing a domain D with boundary $\gamma = \partial D$ lies in a plane Π. Physically we think of D as a rigid *lamina* which rolls without slipping along the a straight line L in another plane Π'. We may suppose the line L to be the x-axis of a Cartesian coordinate system (x, y) for the plane Π'. The locus in Π' of a fixed point $O \in D$ defines a (single-valued) bounded (and hence no-monotonic) periodic graph $\gamma' : y = y(x)$ over L . If γ is smooth and O lies in the interior of D, then $\tilde{\gamma}$ will be a smooth simple curve. If $O \in \partial D$, then γ' will have cusps where O meets the x-axis. If, by some contrivance, O lies outside D, then the curve γ' will no longer be a single-valued graph and it will have self-intersections. If the point O lies on the curve, one refers γ' to a *cycloid*. Otherwise to a *trochoid*. Using O as origin we may specify the curve γ by its equation $\mathbf{r} = \mathbf{r}(t)$. We define $r = |\mathbf{r}|$ and $p = \mathbf{r} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward normal of of γ. Thus r is distance from O to the point $\gamma(t)$ and p the perpendicular distance from O to the tangent of the curve γ at the point $\gamma(t)$. The relation $(p\,, r) \in \mathbb{R}^2$ is called the *pedal equation* of γ with respect to the origin O. We also have a relation $(y\,, y') \in \mathbb{R}^2$, associated to the graph γ, where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$.

Simple geometry gives

$$(p\,,r)=\left(y\,,y\sqrt{1+y'^2}\right),\qquad (y,y')=\left(p,\sqrt{\frac{r^2}{p^2}-1}\right). \tag{3.1}$$

If (r,θ) are plane polar coordinates for the plane Π with origin O, then

$$p=\frac{r}{\sqrt{1+\frac{1}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}}\,,\qquad\iff\qquad \left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+u^2=\frac{1}{p^2}. \tag{3.2}$$

Thus given a pedal curve we get a relation (y,y') and if we are fortunate we may be able to solve for $y(x)$. Conversely given the graph $y(x)$ we may determine the relation (p,r) and if we are fortunate we may be able to solve for $p=f(r)$ and hence determine the equation of the pedal curve in in polar coordinates by quadratures. There remains the problem of determining whether the pedal curve is among the known plane curves catalogued in various books, and if so what are its properties.

We conclude this section by recalling Steiner's theorem, which states that

- the arc length along the rolling curve equals the arc length along the line on which it rolls.

The area under the roulette is twice the area swept out by the rolling curve.

3.2 The Friedmann equation

It is convenient in what follows to use units in which $8\pi G_3=3$ and $c=1$, and to rewrite the Friedmann equation as

$$1+(\dot a)^2=a^2\rho(a)+(1-k)a^2. \tag{3.3}$$

Now we may rescale a and t in (3.3) by the same constant factor f and transform the equation to the form

$$1+(\dot a)^2=f^2a^2\rho(fa)+f^2(1-k)a^2. \tag{3.4}$$

It follows from (3.1) that the pedal equation of the roulette we seek is given by

$$r^2=p^4f^2\rho(fp)+f^2(1-k)p^4, \tag{3.5}$$

where f may be chosen as we wish. Setting $k=1$ and $f=1$ gives our basic eqation:

$$\boxed{r^2=p^4\rho(p)\qquad\iff\qquad\rho(p)=\frac{r^2}{p^4}} \tag{3.6}$$

which may be read in two directions. Either we know $\rho(a)$, and we deduce the pedal equation, or given, a pedal equation, we deduce $\rho(a)$. In the sequel we shall proceed in both directions.

Before doing so we recall that in terms of conformal time η, defined by $\mathrm{d}t = a\mathrm{d}\eta$, the Friedmann equation may be written (if $k = 1$) as

$$\left(\frac{\mathrm{d}a}{\mathrm{d}\eta}\right)^2 + ka^2 = a^4\rho(a), \tag{3.7}$$

which is a first integral of a non-linear oscillator equation obtained by differentiating (3.7) with respect to η, which is equivalent to the familiar Raychaudhuri equation.

It is readily seen that the integrable cases in view of the Chebyshev theorem include the model

$$\rho(a) = \alpha a^{-2} + \beta a^{\sigma}, \tag{3.8}$$

where α, β, σ are arbitrary constants.

If we introduce [18] the inverse scale factor, or redshift factor by

$$b = \frac{1}{a}, \tag{3.9}$$

we find

$$\left(\frac{\mathrm{d}b}{\mathrm{d}\eta}\right)^2 + kb^2 = \rho\left(\frac{1}{b}\right). \tag{3.10}$$

3.2.1 Steiner's Theorem

With this preparation we note that Steiner's theorem implies that if we are considering the scale factor as a function of cosmic time, $a = a(t)$, the arc-length along the rolling curve equal to cosmic time and the area $\int a(t)\mathrm{d}t$ swept out is twice the area swept out on the rolling curve. The area under the graph of the scale factor plotted against time has no obvious meaning but if instead we consider scale factor as a function of conformal time, $a = a(\eta)$, which we represented as a roulette, then arc-length of the rolling curve would be equal to cosmic time and the area under the graph $\int a(\eta)\mathrm{d}\eta$ swept out would be equal to cosmic time and as a consequence this would be twice the area swept out by the rolling curve.

3.3 A single-component fluid in a closed universe

In the simplest barotropic case with $k = 1$ and $\gamma \neq \frac{2}{3}$ we may choose the scaling factor f so that

$$\rho = \frac{1}{a^{3\gamma}}. \tag{3.11}$$

In fact John Barrow has pointed out [19] that in this case, if one defines

$$g = a^{\frac{3\gamma-2}{2}}, \tag{3.12}$$

then the Friedmann equation becomes

$$\left(\frac{\mathrm{d}g}{\mathrm{d}\eta}\right)^2 + \left(\frac{3\gamma-2}{2}\right)^2 g^2 = \left(\frac{3\gamma-2}{2}\right)^2, \tag{3.13}$$

whence we obtain the Raychaudhuri equation

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\eta^2} + \left(\frac{3\gamma - 2}{2}\right)^2 g = 0, \tag{3.14}$$

and so a general solution for a as a function of conformal time η is available and cosmic time t may then be obtained in principle by quadratures.

Not surprisingly, in view of the previous subsection and the above discussion, the same reduction may be carried out in the context of the slightly more general model (3.8) with $\sigma = -3\gamma$ and $\gamma \neq \frac{2}{3}$, which leads to the updated equation

$$\left(\frac{\mathrm{d}g}{\mathrm{d}\eta}\right)^2 + (k - \alpha)\left(\frac{3\gamma - 2}{2}\right)^2 g^2 = \beta \left(\frac{3\gamma - 2}{2}\right)^2, \tag{3.15}$$

and may be solved as before.

However we wish to construct the solution as a roulette. The pedal equation is seen to be

$$r^2 = p^{4-3\gamma}, \tag{3.16}$$

which is the pedal equation of the curve

$$r^{\frac{3\gamma-2}{4-3\gamma}} = \cos\left(\frac{3\gamma - 2}{4 - 3\gamma}\theta\right). \tag{3.17}$$

Note that the $\gamma = \frac{2}{3}$ is a critical case separating solutions which oscillate in conformal time and those that blow up exponentially. Another special case is $\gamma = \frac{4}{3}$, radiation. Both require special treatment.

Some special cases are as follows.

- For dust, $\gamma = 1$, $P = 0$, we have

$$r = \cos\theta\,, \tag{3.18}$$

  which is the polar equation of a circle of radius $\frac{1}{2}$, where the origin is on its circumference.

- If $\gamma = \frac{10}{9}$, $P = \frac{1}{9}\rho$, we have

$$r = \cos(2\theta), \tag{3.19}$$

  which is the polar equation of the Lemniscate of Bernoulli whose Cartesian equation is

$$(x^2 + y^2)^2 = x^2 - y^2. \tag{3.20}$$

- For radiation, $\gamma = \frac{4}{3}$, $P = \frac{1}{3}$, we have

$$r = 1, \tag{3.21}$$

which is which is the polar equation of a circle of radius 1 , where the origin is at its centre. However the locus of such a rolling curve would be a straight horizontal line. This is not consistent with the second-order Raychaudhuri equation (3.14) whose solution is

$$a = \sin\eta, \qquad t = 1 - \cos\eta. \tag{3.22}$$

- For stiff matter, $\gamma = 2$, $P = \rho$, we have

$$r^2 \cos(2\theta) = 1, \tag{3.23}$$

  which is the rectangular hyperbola

$$x^2 - y^2 = 1, \tag{3.24}$$

  with the origin at its centre $(x, y) = (0, 0)$.

- If $\gamma = \frac{8}{9}$, $P = -\frac{1}{9}\rho$, we have

$$r = \frac{1}{2}(1 + \cos\theta), \tag{3.25}$$

  which is the polar equation of a cardioid with origin at the cusp $\theta = \pi$.

- If $\gamma = \frac{5}{6}$, $P = -\frac{1}{6}\rho$, we have

$$r = \cos^3\left(\frac{\theta}{3}\right), \tag{3.26}$$

  which is the polar equation of Cayley's sextic whose Cartesian equation is

$$\left(4(x^2 + y^2) - x\right)^3 = 27\left(x^2 + y^2\right)^2. \tag{3.27}$$

- If $\gamma = \frac{1}{3}$, $P = -\frac{2}{3}\rho$, we have

$$r = \cos^{-3}\left(\frac{\theta}{3}\right), \tag{3.28}$$

  which is the polar equation of Tschirhausen's cubic, whose Cartesian equation is

$$27y^2 = (1 - x)(x + 8)^2. \tag{3.29}$$

  Tschirhausen's cubic is also known as L'Hôpital's cubic or Catalan's trisectrix.

- For a cosmological term, $\gamma = 0$,

$$\frac{2}{r} = 1 + \cos\theta, \tag{3.30}$$

  which is the polar equation of a parabola of semi-latus rectum 2 and origin at its focus.

We now turn to the special case $\gamma = \frac{2}{3}$, that is $P = -\frac{1}{3}\rho$. To begin with we assume $k = 1$. Now from (3.6) or (3.3) $\rho_0 \geq 1$, the pedal equation becomes

$$r^2 = p^2 \frac{1}{\sin^2 \alpha}, \tag{3.31}$$

for some constant α which in polar coordinates becomes

$$r = a_0 \, \mathrm{e}^{\cot \alpha \theta}, \tag{3.32}$$

where a_0 is a constant. This is a logarithmic spiral unless $\alpha = \frac{\pi}{2}$, in which case it is a circle of radius a_0. Since Friedmann's equation reads in this case

$$(\dot{a})^2 = \cot^2 \alpha, \tag{3.33}$$

we see that the scale factor increases or decreases linearly with time unless $\alpha = \frac{\pi}{2}$ in which it is constant.

3.4 A two-component fluid in a closed universe

For our next example, consider the locus of a point at a distance A from the centre of a circle of radius R. One has

$$t = R\eta - A \sin \eta, \tag{3.34}$$
$$a = R - A \cos \eta. \tag{3.35}$$

Note that $\mathrm{d}t = a\mathrm{d}\eta$ and thus η is conformal time. Moreover

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \dot{a} = \frac{R \sin \eta}{R - A \cos \eta}. \tag{3.36}$$

One thus has

$$1 + \dot{a}^2 = \frac{A^2 - R^2 + 2Ra}{a^2}. \tag{3.37}$$

The pedal equation works out to be

$$r^2 = A^2 - R^2 + 2Rp, \tag{3.38}$$

which may also be verified by a simple geometric argument. Comparing (3.37) with (3.6) we find that

$$\rho = \frac{2R}{a^3} + \frac{A^2 - R^2}{a^4}, \tag{3.39}$$

which, if $A > R$ is a mixture of dust and radiation with positive energy density. The model starts out from a big bang, a increases to a maximum, and the model ends with a big crunch. This feature may not be so obvious since the time variable transformation (3.34) is not invertible when $A > R$. However, in [2], we have carried out a systematic study

which shows that the Friedmann equation (3.37) is integrable, in view of the Chebyshev theorem, to yield a big-bang type periodic solution in cosmic time t.

Geometrically we have a *prolate cycloid*. If $A < R$, we have a *curtate cycloid*. The energy density of the radiation is negative and this gives rise to a bounce and hence oscillatory behaviour. This periodic behaviour is clearly seen in (3.35) by virtue of the invertible transformation (3.34) when $R > A$ but was untouched in [2].

A more general example is given by the pedal equation of an epi- or hypoycloid

$$r^2 = \frac{p^2}{C} + A^2, \tag{3.40}$$

with

$$C = \frac{(A+2B)^2}{4B(A+B)}, \tag{3.41}$$

whose parametric equation in Cartesian coordinates is

$$x(\beta) = (A+B)\cos\beta - B\cos\left(\frac{A+B}{B}\beta\right), \tag{3.42}$$

$$y(\beta) = (A+B)\sin\beta - B\sin\left(\frac{A+B}{B}\beta\right). \tag{3.43}$$

Comparing with (3.6), we see that

$$a^4\rho + (k-1)a^2 = \frac{a^2}{C} + A^2, \tag{3.44}$$

whence we have

$$\rho = \left(\frac{1}{C} - (1-k)\right)\frac{1}{a^2} + \frac{A^2}{a^4}. \tag{3.45}$$

The last term in (3.45) corresponds to radiation with a positive pressure, $\gamma = \frac{4}{3}$. The first term to a fluid with $\gamma = \frac{2}{3}$. For particular ratios A/B and hence by (3.41) certain values of CC we have the following special cases

- $C = \frac{9}{8}$, the Cardoid, $A/B = 1$.
- $C = \frac{4}{3}$, the Nephroid, $A/B = 2$.
- $C = -\frac{1}{8}$, the Deltoid, $A/B = -3$.
- $C = -\frac{1}{3}$, the Astroid, $A/B = -4$.

If $k = 0$ or $k = 1$, the first two cases correspond to a fluid with $\gamma = \frac{2}{3}$ with positive energy density and negative pressure, the other cases to negative energy density and positive pressure.

Some other cases are as follows.

- $$r^2 = \frac{A^2B^2}{p^2} + A^2 - B^2 \tag{3.46}$$

  corresponds to the hyperbola
  $$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1, \tag{3.47}$$
  with origin at its centre. If $k = 1$ this has stiff matter $\gamma = 2$ with positive energy density and radiation whose energy density will be positive if $A^2 > B^2$ and negative if $A^2 < B^2$.

- $$r^2 = -\frac{A^2B^2}{p^2} + A^2 + B^2 \tag{3.48}$$

  corresponds to the ellipse
  $$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1, \tag{3.49}$$
  with origin at its centre. If $k = 1$ this has stiff matter $\gamma = 2$ with negative energy density, in other words, a phantom, and radiation with positive energy density.

- $$r^2 = \left(\frac{4A^2B^2}{p^2} + 4A^2\right)^{-2}. \tag{3.50}$$

  This is an ellipse with origin at a focus.
  $$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1. \tag{3.51}$$
  If $k = 1$, the energy density is
  $$\rho = \frac{1}{(2AB)^4}\frac{1}{(1+a^2)^2}. \tag{3.52}$$
  The curve pursued by the scale factor is called an elliptic catenary.

- $$r^2 = \left(\frac{4A^2B^2}{p^2} - 4A^2\right)^{-2}. \tag{3.53}$$

  This is an hyperbola *but now with origin at a focus.*
  $$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1. \tag{3.54}$$
  If $k = 1$, the energy density is
  $$\rho = \frac{1}{(2AB)^4}\frac{1}{(1-a^2)^2}. \tag{3.55}$$
  The curve pursued by the scale factor is called a hyperbolic catenary.

3.5 ΛCDM cosmology

The presently favoured, so-called concordance model has $k = 0$ and

$$\rho = \frac{1}{a^3} + \frac{\Lambda}{3}, \tag{3.56}$$

corresponding to dust which is known to be integrable.

The pedal equation is therefore

$$\begin{aligned} r &= p\sqrt{p^2\left(\frac{1}{p^3} + \frac{\Lambda}{3} + \frac{1}{p^2}\right)} && (3.57) \\ &= \sqrt{p + \frac{\Lambda}{3}p^4 + p^2}. && (3.58) \end{aligned}$$

There is a simple formula for the scale factor as a function of time but finding the curve of which it is the roulette appears to be difficult since one has to solve a quartic curve for p.

The reader will have noticed that many of the explicit examples of roulettes giving rise to solutions of Friedmann's equations descibed above have involved "exotic equations of state" with components having for example negative energy densities and/or pressures exceeding energy densities in magnitude. These were often excluded from traditional texts on cosmology on the grounds of lack of physical interest. However with the growing observational support for the presence of dark energy the literature has become a great deal less inhibited (see e.g. [20]), and we have therefore felt justified in including as many explicit cases of the roulette construction as we could find.

4 Quadratic equation of state

It is well known that the flat-universe ($k = 0$) Friedmann equation in the linear equation of state case

$$P = A\rho, \tag{4.1}$$

where A is a constant satisfying the non-phantom condition $A > -1$, may be integrated for arbitrary values of the cosmological constant Λ to allow big-bang solutions with $a(0) = 0$, so that when $\Lambda = 0$ the scale factor $a(t)$ grows following a power law, when $\Lambda > 0$, an exponential law (a dark energy regime), and, when $\Lambda < 0$ the scale factor $a(t)$ oscillates periodically.

In this section, we consider the more general situation [7, 8, 9] when the equation of state follows the quadratic law

$$P = P_0 + A\rho + B\rho^2, \tag{4.2}$$

which may be viewed as a second-order truncation approximation of the general equation of state, $P = P(\rho)$. We note that cosmologies governed by quadratic and some other more extended forms of equations of state were investigated earlier in [21, 22, 23]. For convenience, we again assume non-phantom condition, $A > -1$, throughout our study.

4.1 Restrictions to parameters in non-degenerate situations

Inserting (4.2) into the law of energy conservation,

$$\dot{\rho} + n(\rho + P)\frac{\dot{a}}{a} = 0, \tag{4.3}$$

we obtain

$$J \equiv \int \frac{\mathrm{d}\rho}{P_0 + (1+A)\rho + B\rho^2} = \ln(Ca^{-n}), \tag{4.4}$$

where $C > 0$ is a constant.

For convenience, use D to denote the discriminant of the quadratic

$$Q(\rho) = P_0 + (1+A)\rho + B\rho^2. \tag{4.5}$$

That is, $D = (1+A)^2 - 4BP_0$.

For $B \neq 0$, there are three subcases to consider.

(i) $D = 0$. Then $J = -\frac{1}{B(\rho-\rho_0)}$ where $\rho_0 = -\frac{1+A}{2B}$. Hence, absorbing a possible integrating constant, we have

$$-\frac{1}{B(\rho - \rho_0)} = \ln(Ca^{-n}). \tag{4.6}$$

We are interested in a scale factor a initiating from sufficiently small values and evolving into arbitrarily large values ('big bang cosmology'). If $B > 0$, then $-\rho > 0$. Using this fact and $\rho \geq 0$ in (4.6), we see that small values of a are not allowed. If $B < 0$, then (4.6) assumes the form

$$\frac{1}{|B|\rho - \frac{1}{2}(1+A)} = \ln(Ca^{-n}). \tag{4.7}$$

To allow a to be small, we obtain the restriction $|B|\rho - \frac{1}{2}(1+A) > 0$, which prohibits a to assume large values. In conclusion, the case $D = 0$ is ruled out.

(ii) $D > 0$. Then $Q(\rho)$ has two roots:

$$\rho_{1,2} = \frac{-(1+A) \pm \sqrt{D}}{2B}, \tag{4.8}$$

resulting in $J = D^{-\frac{1}{2}} \ln\left|\frac{\rho-\rho_1}{\rho-\rho_2}\right|$. Hence

$$\rho = \frac{\rho_1 \pm Ca^{-n\sqrt{D}}\rho_2}{1 \pm Ca^{-n\sqrt{D}}}, \tag{4.9}$$

where $C > 0$ is an updated constant. Inserting the epoch $a = 0$ into (4.9), we get

$$\rho|_{a=0} = \rho_0 = \rho_2 = \frac{1 + A + \sqrt{D}}{2|B|} > 0, \quad B < 0. \tag{4.10}$$

In other words we find a sign restriction for B. Inserting $a(\infty)=\infty$ into (4.9), we get

$$\rho(\infty)=\rho_1=\frac{1}{2|B|}([1+A]-\sqrt{[1+A]^2+4|B|P_0})\geq 0. \tag{4.11}$$

Therefore we must have

$$P_0\leq 0. \tag{4.12}$$

Using $\rho_1<\rho(t)<\rho_2$ $(\forall t)$, we see that (4.9) is made precise:

$$\rho=\frac{\rho_1 a^{n\sqrt{D}}+C\rho_2}{a^{n\sqrt{D}}+C}. \tag{4.13}$$

(iii) $D<0$. Then $Q(\rho)$ has no real roots. We can integrate (4.4) to get

$$\frac{2B}{\sqrt{-D}|B|}\arctan\frac{|B|}{\sqrt{-D}}\left(2\rho+\frac{1+A}{B}\right)=\ln(Ca^{-n}). \tag{4.14}$$

Since the right-hand side of (4.14) stays bounded, we see that small values of a are not allowed. Thus the case $D<0$ is also ruled out.

4.2 Global solutions

Hence we now aim at integrating the flat-space ($k=0$) Friedmann equation

$$\left(\frac{\dot a}{a}\right)^2=\frac{16\pi G_n}{n(n-1)}\rho+\frac{2\Lambda}{n(n-1)} \tag{4.15}$$

in the case $D>0$ only. For this purpose, we may insert (4.13) into (4.15) to get

$$\left(\frac{\dot a}{a}\right)^2=\frac{L_1 a^{n\sqrt{D}}+L_2}{a^{n\sqrt{D}}+C}, \tag{4.16}$$

where

$$L_1=\frac{2}{n(n-1)}(8\pi G_n\rho_1+\Lambda),\quad L_2=\frac{2C}{n(n-1)}(8\pi G_n\rho_2+\Lambda). \tag{4.17}$$

Thus, an integration of (4.16) gives us

$$\begin{aligned} t &= I\equiv\int\sqrt{\frac{a^{n\sqrt{D}}+C}{L_1 a^{n\sqrt{D}}+L_2}}\,\frac{\mathrm{d}a}{a} \\ &= \frac{1}{n\sqrt{D}}\int\sqrt{\frac{u+C}{L_1 u+L_2}}\,\frac{\mathrm{d}u}{u}\quad (u=a^{n\sqrt{D}}), \end{aligned} \tag{4.18}$$

for which the Chebyshev theorem is not applicable in general.

If $L_1 = 0$, then $L_2 \neq 0$ otherwise $\rho_1 = \rho_2$ and $D = 0$ which is false. Thus $L_2 > L_1 = 0$. In this case we can set $u = L_2 v^2 - C$ to get

$$\begin{aligned} t &= I = \frac{2}{n\sqrt{D}} \int \frac{L_2 v^2}{L_2 v^2 - C} \, \mathrm{d}v \\ &= \frac{2}{n\sqrt{D}} v + \frac{1}{n\sqrt{D}} \sqrt{\frac{C}{L_2}} \ln \left| \frac{v - \sqrt{\frac{C}{L_2}}}{v + \sqrt{\frac{C}{L_2}}} \right| + C_1, \end{aligned} \tag{4.19}$$

where C_1 is an integration constant and

$$v = \sqrt{\frac{a^{n\sqrt{D}} + C}{L_2}}. \tag{4.20}$$

In other words, (4.19) and (4.20) give us the general solution of the Friedmann equation when $L_1 = 0$.

From this solution we see that the initial condition $a = 0$, or

$$v = \sqrt{\frac{C}{L_2}}, \tag{4.21}$$

cannot be achieved at any finite time but can only be realized at $t = -\infty$. More precisely, from (4.19) and (4.20), we may deduce the property

$$a(t) = \mathrm{O}\left(\mathrm{e}^{\sqrt{\frac{L_2}{C}}\, t} \right), \quad t \to -\infty. \tag{4.22}$$

Furthermore, from (4.19) and (4.20), we may also deduce the asymptotic behavior

$$a(t) = \mathrm{O}\left(t^{\frac{2}{n\sqrt{D}}} \right), \quad t \to \infty. \tag{4.23}$$

Now assume $L_1 \neq 0$. From (4.16) we have $L_1, L_2 \geq 0$. Hence $L_1 > 0$. Thus $L_2 > L_1 > 0$. In this case we can set

$$v = \sqrt{\frac{u + C}{L_1 u + L_2}} \quad \text{or} \quad u = -\frac{L_2 v^2 - C}{L_1 v^2 - 1} = -\frac{\left(\frac{L_2}{L_1} - C\right)}{L_1 v^2 - 1} - \frac{L_2}{L_1}. \tag{4.24}$$

Consequently, we obtain

$$\begin{aligned} t &= I = \frac{2(CL_1 - L_2)}{n\sqrt{D}} \int \frac{v^2}{(L_1 v^2 - 1)(L_2 v^2 - C)} \, \mathrm{d}v \\ &= \frac{1}{n\sqrt{D}} \left(\frac{1}{\sqrt{L_1}} \ln \left| \frac{v + \frac{1}{\sqrt{L_1}}}{v - \frac{1}{\sqrt{L_1}}} \right| + \sqrt{\frac{C}{L_2}} \ln \left| \frac{v - \sqrt{\frac{C}{L_2}}}{v + \sqrt{\frac{C}{L_2}}} \right| \right) + C_1, \end{aligned} \tag{4.25}$$

where C_1 is an integration constant. Since

$$v = \sqrt{\frac{a^{n\sqrt{D}} + C}{L_1 a^{n\sqrt{D}} + L_2}}, \tag{4.26}$$

we see that $a = 0$ corresponds to $v = \sqrt{\frac{C}{L_2}}$ like before.

From (4.17), we have

$$\frac{L_2}{C} = \frac{2}{n(n-1)}(8\pi G_n \rho_2 + \Lambda) > \frac{2}{n(n-1)}(8\pi G_n \rho_1 + \Lambda) = L_1, \tag{4.27}$$

which leads to $v > \frac{1}{\sqrt{L_1}}$. Consequently we see in view of (4.25) and (4.26) that the scale factor a cannot vanish at any finite time but at $t = -\infty$. More precisely, we have

$$a(t) = \mathrm{O}\left(\mathrm{e}^{\sqrt{\frac{L_2}{C}}\, t}\right), \quad t \to -\infty, \tag{4.28}$$

which is the same as (4.22) and may hardly be surprising. Similarly, we have

$$a(t) = \mathrm{O}\left(\mathrm{e}^{\sqrt{L_1}\, t}\right), \quad t \to \infty. \tag{4.29}$$

This is an important conclusion since we have achieved an exponential asymptotic growth law. It is worth noting that the growth rate $\sqrt{L_1}$ is given by the explicit formula

$$\begin{aligned} L_1 &= \frac{2}{n(n-1)}\left(\frac{4\pi G_n}{|B|}\left[(1+A) - \sqrt{(1+A)^2 - 4|BP_0|}\right] + \Lambda\right), \\ & B < 0, \quad P_0 \leq 0. \end{aligned} \tag{4.30}$$

In summary, we can conclude:

(i) A nontrivial quadratic equation of state given in (4.2) dictates the necessary and sufficient condition

$$\begin{aligned} & P_0 \leq 0, \quad B < 0, \quad (1+A)^2 + 4|B|P_0 > 0, \\ & \frac{4\pi G_n}{|B|}\left((1+A) - \sqrt{(1+A)^2 - 4|BP_0|}\right) + \Lambda \geq 0, \end{aligned} \tag{4.31}$$

in order to allow the scale factor a to evolve from the epoch $a = 0$ to the epoch $a = \infty$.

(ii) The quadratic equation of state (4.2) may never allow a finite initial time at which the scale factor vanishes.

(iii) In the critical situation

$$\frac{4\pi G_n}{|B|}\left((1+A)-\sqrt{(1+A)^2-4|BP_0|}\right)+\Lambda=0, \tag{4.32}$$

the scale factor a grows following a power law. On the other hand, in the non-critical situation

$$\frac{4\pi G_n}{|B|}\left((1+A)-\sqrt{(1+A)^2-4|BP_0|}\right)+\Lambda>0, \tag{4.33}$$

however, the scale factor a obeys the exponential growth law (4.29)–(4.30). In particular, in this latter situation, the cosmological constant Λ is permitted to assume a suitable negative value when $P_0<0$.

We may compare our results with the α-fluid model studied in [24] for which the equation of state is

$$P=-(\alpha+1)\rho_\Lambda+\alpha\rho-(\alpha+1)\frac{\rho^2}{\rho_P}, \tag{4.34}$$

where $\alpha\geq 0$ is a parameter, ρ_P is the Planck density, and ρ_Λ the cosmological density. It is seen that (4.34) is consistent with a part of the necessary and sufficient condition just derived. Furthermore, the positivity condition for the discriminant of (4.5), i.e., $D>0$, translates itself into the form

$$\frac{\rho_\Lambda}{\rho_P}<\frac{1}{4}, \tag{4.35}$$

which does not involve α, in particular. Since ρ_Λ is of order 10^{-24} and ρ_P of order 10^{99} (cf. [24]), the condition (4.35) is well observed too.

4.3 Degenerate situation

In the degenerate situation, $B=0$, (4.4) gives us the relation

$$|P_0+(1+A)\rho|=C^{1+A}a^{-n(1+A)}. \tag{4.36}$$

Since we are interested in the case when a may assume arbitrarily large values as $t\to\infty$ and $\rho\geq 0$, we again arrive at the necessary condition

$$P_0\leq 0. \tag{4.37}$$

Hence

$$\rho=\frac{1}{1+A}\left(C^{1+A}a^{-n(1+A)}+|P_0|\right). \tag{4.38}$$

In other words, the presence of P_0 simply and unsurprisingly adds a shift to the cosmological constant.

4.4 Polytropic equation of state case

It will also be interesting to extend our study in this section into a slightly more general situation where the equation of state is of a polytropic one given by [25, 26]

$$P = A\rho + \kappa\rho^{\gamma}, \quad A > -1, \quad \gamma \neq 1, \quad \gamma \geq 0, \tag{4.39}$$

where $\kappa \neq 0$ otherwise the equation of state becomes the well-studied linear case.

Inserting (4.39) into the equation of energy conservation, we have

$$(1 + A)\rho^{1-\gamma} + \kappa = Ca^{-n(1+A)(1-\gamma)}, \tag{4.40}$$

where $C > 0$ is an integration constant.

(i) $\gamma < 1$. Then (4.40) indicates that the epoch $a = 0$ gives rise to the initial value $\rho_0 = \infty$ and the epoch $a = \infty$ gives rise to the condition $\kappa < 0$ and the limiting value

$$\rho_\infty = \left(\frac{|\kappa|}{1 + A}\right)^{\frac{1}{1-\gamma}}. \tag{4.41}$$

(ii) $\gamma > 1$. Then (4.40) implies that the epoch $a = 0$ gives rise to the condition $\kappa < 0$ again and the initial value

$$\rho_0 = \left(\frac{|\kappa|}{1 + A}\right)^{\frac{1}{1-\gamma}}, \tag{4.42}$$

and the epoch $a = \infty$ gives rise to the limiting value $\rho_\infty = 0$.

Note that the necessary condition $\kappa < 0$ is analogous to the necessary condition $B < 0$ stated in (4.31) for the quadratic equation of state problem.

Applying (4.40) in the flat-space Friedmann equation with $\Lambda = 0$ for simplicity, we arrive at the integral

$$4\left(\frac{\pi G_n}{n(n-1)}\right)^{\frac{1}{2}} (1 + A)^{-\frac{1}{2(1-\gamma)}}\, t = I = \int a^{-1}\left(Ca^{-n(1+A)(1-\gamma)} + |\kappa|\right)^{-\frac{1}{2(1-\gamma)}} \,\mathrm{d}a. \tag{4.43}$$

In view of the Chebyshev theorem, we know that the integral I is elementary when A and γ are arbitrary rational numbers. Here we omit presenting the solutions.

5 The Randall–Sundrum II cosmology

In this section we consider the braneworld cosmological Friedmann equation [10]

$$H^2 + \frac{k}{a^2} = \alpha_1\rho + \alpha_2\rho^2 + \lambda \tag{5.1}$$

in the Randall–Sundrum II universe [11, 12, 13] where

$$\alpha_1 = \frac{16\pi G_n}{n(n-1)}, \quad \alpha_2 = \left(\frac{16\pi G_{n+1}}{n}\right)^2, \quad \lambda = \frac{2\Lambda}{n(n-1)} \tag{5.2}$$

are specific physical parameters.

5.1 Linear equation of state case

In view of the linear equation of state

$$P = A\rho \tag{5.3}$$

and the law of energy conservation (4.3), we have

$$\rho = Ca^{-n(1+A)}, \quad 1 + A > 0, \quad C > 0. \tag{5.4}$$

Inserting (5.4) into (5.1) and restricting to the flat-space case $k = 0$, we easily obtain the integration

$$\begin{aligned} t + C_1 &= I = \int \frac{\mathrm{d}a}{a\sqrt{\beta_1 a^{-n(1+A)} + \beta_2 a^{-2n(1+A)} + \lambda}} \\ &= \int \frac{a^{n(1+A)-1}\mathrm{d}a}{\sqrt{\lambda a^{2n(1+A)} + \beta_1 a^{n(1+A)} + \beta_2}} \\ &= \frac{1}{n(1+A)} \int \frac{\mathrm{d}u}{\sqrt{\lambda u^2 + \beta_1 u + \beta_2}} \qquad (u = a^{n(1+A)}) \\ &= \begin{cases} \frac{2}{n(1+A)\beta_1}\sqrt{\beta_1 a^{n(1+A)} + \beta_2}, & \lambda = 0, \\ \frac{1}{n(1+A)\sqrt{\lambda}} \ln\left(\frac{\beta_1}{2\lambda} + a^{n(1+A)} + \sqrt{a^{2n(1+A)} + \frac{\beta_1}{\lambda}a^{n(1+A)} + \frac{\beta_2}{\lambda}}\right), & \lambda > 0, \\ \frac{1}{n(1+A)\sqrt{|\lambda|}} \arctan\left(\frac{a^{n(1+A)} - \frac{\beta_1}{2|\lambda|}}{\sqrt{-a^{2n(1+A)} + \frac{1}{|\lambda|}(\beta_1 a^{n(1+A)} + \beta_2)}}\right), & \lambda < 0, \end{cases} \end{aligned} \tag{5.5}$$

since it is of the Chebyshev form, where $\beta_1 = C\alpha_1$ and $\beta_2 = C^2\alpha_2$ and C_1 is an integration constant which may always be chosen to allow the big-bang initial condition $a(0) = 0$. Indeed, from (5.5), we have

$$C_1 = \begin{cases} \frac{2\sqrt{\beta_2}}{n(1+A)\beta_2}, & \lambda = 0, \\ \frac{1}{n(1+A)\sqrt{\lambda}} \ln\left(\frac{\beta_1}{2\lambda} + \sqrt{\frac{\beta_2}{\lambda}}\right), & \lambda > 0, \\ -\frac{1}{n(1+A)\sqrt{|\lambda|}} \arctan\left(\frac{\beta_1}{2\sqrt{|\lambda|\beta_2}}\right), & \lambda < 0. \end{cases} \tag{5.6}$$

From the above results we deduce, as $t \to \infty$, the growth laws

$$a(t) = \mathrm{O}\left(t^{\frac{2}{n(1+A)}}\right), \quad \lambda = 0; \quad a(t) = \mathrm{O}\left(\mathrm{e}^{\sqrt{\lambda}t}\right), \quad \lambda > 0. \tag{5.7}$$

These two situations are analogous to those in the classical situation [1, 27].

However, it may be a surprise to see, when $\lambda < 0$, the solution will be of a finite lifespan or periodic, instead, depending on how far the parameter A is away from the

phantom divide line [28, 29, 30, 31, 32], $A = -1$. To see this fact more transparently, we rewrite the Friedmann equation as

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 = n^2(1+A)^2|\lambda|(u_1 - u)(u + u_2), \tag{5.8}$$

where

$$u_1 = \frac{1}{2|\lambda|}\left(\beta_1 + \sqrt{\beta_1^2 + 4\beta_2|\lambda|}\right), \quad u_2 = \frac{1}{2|\lambda|}\left(\sqrt{\beta_1^2 + 4\beta_2|\lambda|} - \beta_1\right), \tag{5.9}$$

and $u = a^{n(1+A)}$. Using $u(0) = 0$, we solve

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \alpha\sqrt{(u_1 - u)(u + u_2)}, \quad \alpha = n(1+A)\sqrt{|\lambda|}, \quad t > 0, \tag{5.10}$$

to get

$$\alpha t = \arctan\left(\frac{u - \frac{1}{2}(u_1 - u_2)}{\sqrt{(u_1 - u)(u + u_2)}}\right) + \arctan\left(\frac{1}{2}\frac{(u_1 - u_2)}{\sqrt{u_1 u_2}}\right), \quad t > 0. \tag{5.11}$$

The above relation remains valid until $t = t_0 > 0$ (say) when $u = u_1$. Thus

$$t_0 = \frac{1}{\alpha}\left(\frac{\pi}{2} + \arctan\left[\frac{1}{2}\frac{(u_1 - u_2)}{\sqrt{u_1 u_2}}\right]\right). \tag{5.12}$$

Beyond t_0 the equation (5.10) is invalid. Instead, u reaches a 'stagnation' point, where $\dot{u}(t_0) = 0$ and $u(t_0) = u_1$, and starts to descend. Hence it must evolve through another branch of the Friedmann equation,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\alpha\sqrt{(u_1 - u)(u + u_2)}, \quad \alpha = n(1+A)\sqrt{|\lambda|}, \quad t > t_0, \tag{5.13}$$

which may be integrated similarly. Actually, since both (5.10) and (5.13) are autonomous, the solution of (5.13) satisfying $u(t_0) = u_1$ may simply be obtained from that of (5.10) through replacing the variable t with $t \in [0, t_0]$ for the solution of (5.10) by $2t_0 - t$ with $t \in [t_0, 2t_0]$. In particular, $u(2t_0) = 0$.

5.2 Solutions of finite lifespans

We now show that the solution ceases to exist beyond $2t_0$ when

$$A \geq -1 + \frac{1}{n}. \tag{5.14}$$

In fact, from (5.13), we have

$$\lim_{t \to (2t_0)^-} \dot{u}(t) = -\alpha\sqrt{u_1 u_2} < 0. \tag{5.15}$$

From the relation between the scale factor a and the solution u we find

$$\dot{a}(t) = \frac{1}{n(1+A)}(u(t))^{-\left(1-\frac{1}{n(1+A)}\right)}\dot{u}(t), \tag{5.16}$$

so that

$$\lim_{t\to(2t_0)^-}\dot{a}(t) = -\infty, \quad A > -1+\frac{1}{n}; \quad \lim_{t\to(2t_0)^-}\dot{a}(t) = -\alpha\sqrt{u_1u_2} < 0, \quad A = -1+\frac{1}{n}. \tag{5.17}$$

On the other hand, if the solution would exist beyond $2t_0$, we should have $\dot{a}(t) \geq 0$ when t passes $2t_0$ where $a = 0$. Thus (5.17) indicates that $\dot{a}(t)$ would suffer from a discontinuity at $t = 2t_0$. In other words, the solution to the Friedmann equation cannot exist beyond $2t_0$ which may well be called the lifespan of the solution.

If the coupling constant A satisfies

$$A < -1 + \frac{1}{n}, \tag{5.18}$$

then (5.16) implies that $\dot{a}(t) \to 0$ as $t \to (2t_0)^-$. Thus we may obtain the solution of the Friedmann equation over $[2t_0, 4t_0]$ by shifting the solution over $[0, 2t_0]$ obtained above and maintain the continuity of $\dot{a}(t)$ at $t = 2t_0$. Such a procedure can be repeated so that a periodic solution of period $2t_0$ is constructed.

Thus, we conclude that, when the cosmological constant is negative, the solution a of the Friedmann equation evolving from the initial condition $a(0) = 0$ has either a finite lifespan $T = 2t_0$ or is of period $T = 2t_0$, according to whether the coupling constant A fulfills the condition (5.14) or (5.18).

For later convenience, we document the lifespan or period T here explicitly in terms of the original physical parameters:

$$T = \frac{1}{1+A}\sqrt{\frac{n-1}{2n|\Lambda|}}\left(\pi + 2\arctan\left[\frac{G_n}{2G_{n+1}}\sqrt{\frac{n}{2(n-1)|\Lambda|}}\right]\right), \quad \Lambda < 0. \tag{5.19}$$

Thus we have unveiled a peculiar situation, $A \geq -1 + \frac{1}{n}$, when the solution has only a finite lifespan. This situation is in sharp contrast with the classical case with $\Lambda < 0$ that the big-bang solution is periodic for any $A > -1$.

To end this subsection, it will be instructive to check that, subject to the equation of state (5.3), the phantom world is still spelled out by the condition $1 + A < 0$ as in the classical situation. In fact, for $1+A \neq 0$, the energy conservation law gives us the relation $\rho = Ca^{-n(1+A)}$ as before where $C > 0$ is a constant. Inserting this into the Friedmann equation (5.1) with $k = 0, \lambda = 0$, we have the integration

$$t = I = \int \frac{\mathrm{d}a}{a\sqrt{C\alpha_1 a^{-n(1+A)} + C^2\alpha_2 a^{-2n(1+A)}}}. \tag{5.20}$$

Hence the scale factor $a = a(t)$ is implicitly given by

$$\frac{\alpha_1}{C}a^{n(1+A)} + \alpha_2 = \left(\sqrt{\left(\frac{\alpha_1}{C}\right)a^{n(1+A)}(0) + \alpha_2} + \frac{1}{2}n(1+A)\alpha_1 t\right)^2, \quad t > 0. \tag{5.21}$$

Consequently we see that the initial condition $a(0) = 0$ is permitted when $1 + A > 0$ so that $a = a(t)$ grows following the law

$$a(t) = \mathrm{O}\left(t^{\frac{2}{n(1+A)}}\right), \quad t \to \infty. \tag{5.22}$$

If $1 + A < 0$, however, the initial condition $a(0) = 0$ is not permitted and the Big Rip [33, 34] happens at

$$t_0 = \frac{2}{n|1+A|\alpha_1} \sqrt{\left(\frac{\alpha_1}{C}\right) a^{n(1+A)}(0) + \alpha_2}. \tag{5.23}$$

In particular, $A = -1$ still corresponds to the phantom divide line [28, 29, 30, 31, 32] indeed as in the classical situation, as anticipated.

In [35], the following slightly more general Friedmann equation

$$H^2 = \alpha\rho + \beta\rho^2 + \frac{\gamma}{a^4} \tag{5.24}$$

is considered ($k = 0, \Lambda = 0$), where $\alpha, \beta, \gamma > 0$ are constants. Although this equation cannot be integrated, some asymptotic analysis establishes that any solution with $a(\infty) = \infty$ grows according to the law $a(t) = \mathrm{O}\left(t^{\frac{2}{n(1+A)}}\right)$ as $t \to \infty$, as before, subject to the equation of state $P = A\rho$ ($A > -1$).

5.3 Chaplygin fluid case

Consider the equation of state

$$P = A\rho - \frac{B}{\rho}, \quad A > -1, \quad B > 0, \tag{5.25}$$

of a generalized Chaplyin fluid model [36, 37, 38]. Inserting (5.25) into the law of energy conservation, (4.3), we obtain

$$\rho = \left(Ca^{-2n(1+A)} + \frac{B}{1+A}\right)^{\frac{1}{2}}, \tag{5.26}$$

where $C > 0$ is an integration constant. Substituting (5.26) into the braneworld Friedmann equation (5.1) with $k = 0$ and $\lambda = 0$ and integrating, we arrive at

$$t = I \equiv \int \left(\alpha_1 \left[Ca^{-2n(1+A)} + b\right]^{\frac{1}{2}} + \alpha_2 \left[Ca^{-2n(1+A)} + b\right]\right)^{-\frac{1}{2}} \frac{\mathrm{d}a}{a}, \quad b = \frac{B}{1+A}. \tag{5.27}$$

To proceed, we note that (5.26) gives us

$$a = C^{\frac{1}{2n(1+A)}} (\rho^2 - b)^{-\frac{1}{2n(1+A)}}, \quad \rho^2 > b. \tag{5.28}$$

Combining (5.27) and (5.28), we have

$$I = -\frac{1}{n(1+A)} \int \frac{1}{(\rho^2 - b)} \sqrt{\frac{\rho}{\alpha_1 + \alpha_2\rho}}\, \mathrm{d}\rho, \tag{5.29}$$

which is not of the Chebyshev type. Nevertheless, set

$$u=\sqrt{\frac{\rho}{\alpha_1+\alpha_2\rho}}. \tag{5.30}$$

Then $0\leq u<\frac{1}{\sqrt{\alpha_2}}$ and

$$\rho=\frac{\alpha_1 u^2}{1-\alpha_2 u^2}, \tag{5.31}$$

which renders the integral (5.29) into a rational one,

$$I=-\frac{2}{\alpha_1 n(1+A)}\int\frac{u^2}{u^4-b\alpha_1^{-2}(1-\alpha_2 u^2)^2}\,\mathrm{d}u, \tag{5.32}$$

which can be integrated separately in the parameter regimes

$$\alpha_2>\frac{\alpha_1}{\sqrt{b}},\quad \alpha_2=\frac{\alpha_1}{\sqrt{b}},\quad \alpha_2<\frac{\alpha_1}{\sqrt{b}}, \tag{5.33}$$

respectively.

We begin by displaying the result in the first regime as follows:

$$\begin{aligned} I &= \frac{1}{n(1+A)\sqrt{b}}\left\{\left(\alpha_2+\frac{\alpha_1}{\sqrt{b}}\right)^{-\frac{1}{2}}\operatorname{arctanh}\left(\left(\alpha_2+\frac{\alpha_1}{\sqrt{b}}\right)^{\frac{1}{2}}u\right)\right.\\ &\quad \left.-\left(\alpha_2-\frac{\alpha_1}{\sqrt{b}}\right)^{-\frac{1}{2}}\operatorname{arctanh}\left(\left(\alpha_2-\frac{\alpha_1}{\sqrt{b}}\right)^{\frac{1}{2}}u\right)\right\}+C_1,\quad \alpha_2>\frac{\alpha_1}{\sqrt{b}}, \end{aligned} \tag{5.34}$$

where C_1 is an integration constant. It is clear that we may choose C_1 suitably to allow the big-bang initial condition $a(0)=0$ or $u(0)=\frac{1}{\sqrt{\alpha_2}}$. On the other hand, the growth condition $a(\infty)=\infty$ gives rise to $\rho(\infty)=\sqrt{b}$. Consequently, we arrive at

$$u(\infty)=\left(\alpha_2+\frac{\alpha_1}{\sqrt{b}}\right)^{-\frac{1}{2}}, \tag{5.35}$$

as anticipated. In other words, the first term on the right-hand side of (5.34) dominates as $t\to\infty$, which leads to the following explicit asymptotic estimate:

$$a(t)=\mathrm{O}\left(\mathrm{e}^{\sigma t}\right),\quad t\to\infty, \tag{5.36}$$

where

$$\sigma=\sqrt{b}\left(\alpha_2+\frac{\alpha_1}{\sqrt{b}}\right)^{\frac{1}{2}}=\sqrt{\frac{B}{1+A}}\left(\left[\frac{16\pi G_{n+1}}{n}\right]^2+\frac{16\pi G_n}{n(n-1)}\sqrt{\frac{1+A}{B}}\right)^{\frac{1}{2}}, \tag{5.37}$$

under the condition

$$\alpha_1<\alpha_2\sqrt{b}\quad\text{or}\quad G_n<16\pi\left(1-\frac{1}{n}\right)G_{n+1}^2\sqrt{\frac{B}{1+A}}. \tag{5.38}$$

Likewise, in the third regime, we have

$$
\begin{aligned}
\alpha_1 n(1+A)I &= \frac{\alpha_1}{\sqrt{b}}\left(\frac{\alpha_1}{\sqrt{b}}+\alpha_2\right)^{-\frac{1}{2}}\operatorname{arctanh}\left(\left(\frac{\alpha_1}{\sqrt{b}}+\alpha_2\right)^{\frac{1}{2}}u\right)\\
&\quad -\frac{\alpha_1}{\sqrt{b}}\left(\frac{\alpha_1}{\sqrt{b}}-\alpha_2\right)^{-\frac{1}{2}}\arctan\left(\left(\frac{\alpha_1}{\sqrt{b}}-\alpha_2\right)^{\frac{1}{2}}u\right)+C_1,\\
&\quad \text{where } \alpha_2<\frac{\alpha_1}{\sqrt{b}} \quad \text{or } 16\pi\left(1-\frac{1}{n}\right)G_{n+1}^2\sqrt{\frac{B}{1+A}}<G_n. \qquad (5.39)
\end{aligned}
$$

With (5.39), we have the same asymptotic state (5.35) and we see that (5.36) and (5.37) still hold.

We omit here the critical case $\sqrt{b}\alpha_2=\alpha_1$.

It is interesting to note the different ways the Newton constants G_n and G_{n+1} and the constants A and B make their contributions to the dark energy, σ, as expressed in (5.37).

It will also be enlightening to compare (5.36)–(5.37) with the formulas [2]

$$
\sigma=4\left(\frac{\pi G_n}{n(n-1)}\right)^{\frac{1}{2}}\left(\frac{B}{1+A}\right)^{\frac{1}{4}}, \tag{5.40}
$$

obtained for the big-bang solution of the classical Friedmann equation

$$
H^2=\frac{16\pi G_n}{n(n-1)}\rho, \tag{5.41}
$$

which may simply be obtained from (5.37) by setting $G_{n+1}=0$.

6 Universal growth law for the extended Chaplygin universe

Recall that the generalized Chaplygin fluid is governed by the equation of state [36, 37, 38]

$$
P=A\rho-\frac{B}{\rho^\alpha},\quad A>-1,\quad B>0,\quad 0\leq\alpha\leq 1, \tag{6.1}
$$

whose cosmological implication, especially its relevance to dark energy problem, has been studied extensively. It is direct to see from (6.1) that there holds the scale factor and energy density relation

$$
\rho^{\alpha+1}=Ca^{-n(1+A)(\alpha+1)}+\frac{B}{1+A}. \tag{6.2}
$$

In view of the Chebyshev theorem, we know that the Friedmann equation with $k=0$ and $\Lambda=0$ is integrable [2] when α is rational. Nevertheless, for any real α, we derived in [2] the following explicit universal growth law:

$$
a(t)\sim C_0\mathrm{e}^{\sigma t},\quad t\to\infty;\quad \sigma=4\left(\frac{\pi G_n}{n(n-1)}\right)^{\frac{1}{2}}\left(\frac{B}{1+A}\right)^{\frac{1}{2(\alpha+1)}}. \tag{6.3}
$$

Inserting (6.3) into (6.2), we have

$$\rho(t) \to \rho_\infty = \left(\frac{B}{1+A}\right)^{\frac{1}{\alpha+1}}, \quad t \to \infty. \tag{6.4}$$

Unsuprisingly, ρ_∞ is the only positive root of the function

$$h(\rho) = P + \rho = \rho + A\rho - \frac{B}{\rho^\alpha}. \tag{6.5}$$

Suggested by the above results, we establish in this section a universal growth law in much extended situations, without resorting to an integration of the Friedmann equation.

6.1 Extended Chaplygin fluids

In [14, 15] and [16], the extended Chaplygin fluid model given by the equation of state

$$P = \sum_{l=1}^{M} A_l \rho^l - \frac{B}{\rho^\alpha} \tag{6.6}$$

is studied in the specific situations when $A_l = A > 0$ $(l = 1, \dots, M)$ and $A_l = \frac{1}{l}$ $(l = 1, \dots, M)$, respectively, by numerical means. Motivated by their work, here we consider the extended Chaplygin model governed by the following general equation of state:

$$P = f(\rho) - \sum_{m=1}^{N} \frac{B_m}{\rho^{\alpha_m}}, \quad \rho > 0, \tag{6.7}$$

where $B_1, \dots, B_N > 0$ and $\alpha_1, \dots, \alpha_N \geq 0$ are constants and f is an analytic function satisfying

$$f(0) = 0; \quad f'(\rho) > -1, \quad \rho > 0; \quad \rho + f(\rho) \to \infty \quad \text{as } \rho \to \infty. \tag{6.8}$$

A direct consequence of the above assumption on f is that the function

$$h(\rho) \equiv \rho + f(\rho) - \sum_{m=1}^{N} \frac{B_m}{\rho^{\alpha_m}} \tag{6.9}$$

has exactly one positive root, say $\rho_\infty > 0$, since $h(\rho)$ strictly increases in $\rho > 0$.

It is clear that our assumption on f, motivated from the results in [2] and discussed above, covers all the examples considered in [14, 15, 16].

6.2 Universal growth law

Rewrite the function h in (6.9) in the form

$$h(\rho) = (\rho - \rho_\infty) h_1(\rho). \tag{6.10}$$

Since $h(\rho) < 0$ for $\rho > 0$ small, we see that $h_1(\rho) > 0$ for all $\rho > 0$. Inserting this into the law of energy conservation (4.3), we have

$$\int_{\rho_0}^{\rho} \frac{\mathrm{d}r}{h_1(r)(r-\rho_\infty)} = \ln\left(\frac{a(t_0)}{a}\right)^n, \tag{6.11}$$

where $\rho_0 = \rho(t_0) > 0$ and t_0 is an initial time when $a(t_0) > 0$ and $\rho = \rho(t), a = a(t)$ with $t \geq t_0$. Note that we are dealing with an expanding universe such that $\rho_0 \geq \rho > \rho_\infty$.

In order to extract information from (6.11), we rewrite the integrand of it as

$$\begin{aligned} \frac{1}{h_1(r)(r-\rho_\infty)} &= \frac{1}{h_1(\rho_\infty)(r-\rho_\infty)} + \frac{1}{r-\rho_\infty}\left(\frac{1}{h_1(r)} - \frac{1}{h_1(\rho_\infty)}\right) \\ &\equiv \frac{1}{h_1(\rho_\infty)(r-\rho_\infty)} + q(r), \quad r > 0, \end{aligned} \tag{6.12}$$

where $q(r)$ is a smooth function in $r > 0$. Inserting (6.12) into (6.11), we arrive at

$$\frac{1}{h_1(\rho_\infty)} \ln|\rho - \rho_\infty| = Q(\rho) + \ln\left(a^{-n}\right), \tag{6.13}$$

where

$$Q(\rho) = \frac{1}{h_1(\rho_\infty)} \ln|\rho_0 - \rho_\infty| + n\ln\left(a(t_0)\right) - \int_{\rho_0}^{\rho} q(r)\,\mathrm{d}r, \quad \rho_\infty < \rho \leq \rho_0, \tag{6.14}$$

which is a bounded function. Combining (6.13) and (6.14), we have

$$\rho = \rho_\infty + \mathrm{e}^{h_1(\rho_\infty)Q(\rho)} a^{-nh_1(\rho_\infty)}. \tag{6.15}$$

Inserting (6.15) into the Friedmann equation (4.15), we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \lambda_0 + \alpha\,\mathrm{e}^{h_1(\rho_\infty)Q(\rho)} a^{-nh_1(\rho_\infty)}, \quad \alpha = \frac{16\pi G_n}{n(n-1)}, \tag{6.16}$$

where we set

$$\lambda_0 = \frac{16\pi G_n}{n(n-1)}\rho_\infty + \frac{2\Lambda}{n(n-1)} > 0. \tag{6.17}$$

In view of (6.16) and (6.17), we obtain the integration

$$\begin{aligned} t &= \int \frac{\mathrm{d}a}{a\sqrt{\lambda_0 + \alpha\,\mathrm{e}^{h_1(\rho_\infty)Q(\rho)} a^{-nh_1(\rho_\infty)}}} \\ &= \int \frac{\mathrm{d}a}{\sqrt{\lambda_0}\,a} + \int \left(\frac{1}{a\sqrt{\lambda_0 + \alpha\,\mathrm{e}^{h_1(\rho_\infty)Q(\rho)} a^{-nh_1(\rho_\infty)}}} - \frac{1}{\sqrt{\lambda_0}\,a}\right)\mathrm{d}a \\ &= \frac{1}{\sqrt{\lambda_0}} \ln a + R(a), \end{aligned} \tag{6.18}$$

where

$$
\begin{aligned}
R(a) &= \int \left(\frac{1}{a\sqrt{\lambda_0 + \alpha\, \mathrm{e}^{h_1(\rho_\infty)Q(\rho)} a^{-nh_1(\rho_\infty)}}} - \frac{1}{\sqrt{\lambda_0}\, a} \right) \mathrm{d}a \\
&= -\frac{1}{2} \int \frac{\alpha \mathrm{e}^{h_1(\rho_\infty)Q(\rho)} a^{-1-nh_1(\rho_\infty)}}{\left(\sqrt{\lambda_0 + \alpha\xi(a) \mathrm{e}^{h_1(\rho_\infty)Q(\rho)} a^{-nh_1(\rho_\infty)}}\right)^3} \,\mathrm{d}a, \quad \xi(a) \in [0,1], \qquad (6.19)
\end{aligned}
$$

which is bounded for a near ∞ since $h_1(\rho_\infty) > 0$. Consequently, in view of (6.18) and (6.19), we deduce the universal growth law for the scale factor a as follows:

$$
a(t) = \mathrm{O}\left(\mathrm{e}^{\sqrt{\lambda_0}t}\right), \quad t \to \infty, \qquad (6.20)
$$

where λ_0 is given in (6.17).

6.3 Examples

It will be enlightening to display a few special examples which allow explicit calculations of ρ_∞, and hence, λ_0.

(i) For the classical generalized Chaplygin model (6.1), we see that ρ_∞ is given by (6.4). Hence

$$
\lambda_0 = \frac{16\pi G_n}{n(n-1)} \left(\frac{B}{1+A} \right)^{\frac{1}{\alpha+1}} + \frac{2\Lambda}{n(n-1)}, \qquad (6.21)
$$

which generalizes the corresponding formula obtained in [2].

(ii) Consider the extended Chaplygin fluid governed by the equation of state

$$
P = A_1\rho + A_3\rho^3 - \frac{B}{\rho}. \qquad (6.22)
$$

It is direct to see that

$$
\rho_\infty = \left(\frac{1}{2A_3} \left[\sqrt{(1+A_1)^2 + 4A_3 B} - (1+A_1) \right] \right)^{\frac{1}{2}}. \qquad (6.23)
$$

Consequently we have

$$
\lambda_0 = \frac{16\pi G_n}{n(n-1)} \left(\frac{2B}{(1+A_1) + \sqrt{(1+A_1)^2 + 4A_3 B}} \right)^{\frac{1}{2}} + \frac{2\Lambda}{n(n-1)}. \qquad (6.24)
$$

In the limit $A_3 \to 0$, (6.24) recovers (6.21) for $\alpha = 1$.

(iii) As another simple example, let the equation of state be

$$P = A\rho - B_0 - \frac{B_1}{\rho}. \tag{6.25}$$

Then we have

$$\rho_\infty = \frac{B_0 + \sqrt{B_0^2 + 4B_1(1+A)}}{2(1+A)}, \tag{6.26}$$

which allows us to return to (6.21) with $\alpha = 1$ again in the limit $B_0 \to 0$.

There are other explicitly solvable cases of interest as well. For example, we consider the equation of state

$$P = A\rho - \frac{B_1}{\rho^{\alpha_1}} - \frac{B_2}{\rho^{\alpha_2}}, \quad \alpha_2 > \alpha_1 \geq 0. \tag{6.27}$$

To solve the associated equation

$$(1+A)\rho - \frac{B_1}{\rho^{\alpha_1}} - \frac{B_2}{\rho^{\alpha_2}} = 0, \tag{6.28}$$

we require $1 + \alpha_2 = 2(\alpha_2 - \alpha_1)$ or $\alpha_2 = 1 + 2\alpha_1$. Hence $\alpha_1 \geq 0$ is allowed to be arbitrary, which contains the case (iii) above as a special example where $\alpha_1 = 0$. In other words, we find an explicit solvable general situation with the equation of state

$$P = A\rho - \frac{B_1}{\rho^{\alpha}} - \frac{B_2}{\rho^{1+2\alpha}}, \quad \alpha \geq 0. \tag{6.29}$$

With (6.29), we obtain ρ_∞ (hence λ_0) explicitly:

$$\rho_\infty = \left(\frac{B_1 + \sqrt{B_1^2 + 4(1+A)B_2}}{2(1+A)}\right)^{\frac{1}{1+\alpha}}. \tag{6.30}$$

Similarly, with

$$P = A_1\rho + A_2\rho^{2+\alpha} - \frac{B}{\rho^{\alpha}}, \quad \alpha \geq 0, \tag{6.31}$$

we have

$$\rho_\infty = \left(\frac{\sqrt{(1+A_1)^2 + 4A_2B} - (1+A_1)}{2A_2}\right)^{\frac{1}{1+\alpha}}, \tag{6.32}$$

which covers (ii) with $\alpha = 1$.

7 Conclusions

Our results in this work are summarized as follows.

(i) *Friedmann equation and roulettes.* We have shown that every solution of Friedmann equation admits a representation as a roulette, that is the locus of a point in the plane of a curve which rolls without slipping on a straight line. A method is given for finding the curve. A well-known example is the scale factor of closed universe containing pressure free matter. The scale factor is a cycloid, i.e. the locus of a point on the circumference of a circle, which rolls without slipping on a straight line. Adding radiation, one finds that curve is still a circle but the point moves outside its circumference. Many other explicit examples are given, some involving exotic equations of state which have been of current interest to cosmologists in connection with dark energy and the pre-inflationary universe.

The Friedmann equation, or its equivalent, also arises in other contexts in physics not directly connected with cosmology such as central orbit problems and geometric optics. Examples of the use of roulettes are in this wider setting are given.

(ii) *Quadratic equation of state.* Although the flat-space Friedmann equation is not in a binomial form so that the Chebyshev theorem is applicable, a necessary and sufficient condition is unveiled for the coefficients of the equation of state to allow the scale factor a to evolve from zero to infinity. A distinguishing characteristic is that the presence of the quadratic term in the equation of state prohibits a to vanish at a finite initial time. In other words, the epoch $a = 0$ may only happen at $t = -\infty$. In the general non-critical situations, a grows exponentially even when the cosmological constant Λ assumes a negative value above a specific critical level.

(iii) *Randall–Sundrum II universe.* In the Randall–Sundrum II universe, there are two interesting flat-space cases: When the equation of state is linear, it is shown that the scale factor a grows following a power law or an exponential law according to whether the cosmological constant Λ is zero or positive, as in the classical situation, but when Λ is negative, however, a new phenomenon occurs so that in an explicitly given regime away from the phantom divide line, the big-bang solution satisfying $a(0) = 0$ has only a finite lifespan, which never happens in the classical situation, although near the phantom divide line the solution is periodic, as in the classical situation; when the equation of state is that of a Chaplygin fluid, we explicitly derive the big-bang solution and describe its exponential growth pattern in terms of various coupling parameters, when $\Lambda = 0$. We see as in the classical Chaplygin fluid model case that a small amount of exotic nonlinear matter would lead to a large amount of presence of dark energy near the phantom divide line, as realized by the exponential growth rate formula.

(iv) *Universal growth formula.* For the extended Chaplygin fluid universe for which the equation of state is decomposed into the sum of a positive analytic function and a negative inverse power function, an integration is impossible in general. To tackle the problem, an analytic method is introduce which enables us to extract all the key parameters that give rise to a universal formula for the exponential growth rate

of the scale factor. Such a formula covers all known formulas derived in concrete situations.

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8 Appendix: Pedal equations and their inversion

If a curve in the plane is given in polar coordinates (r, ϕ) by

$$u = \frac{1}{r} = u(\phi), \tag{8.1}$$

and if p is the perpendicular distance from the origin to the tangent at the point (r, ϕ), one has

$$\frac{1}{p^2} = \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2. \tag{8.2}$$

Conversely, if we are given the pedal equation of a curve in the form

$$\frac{1}{p^2} = P(u), \tag{8.3}$$

the problem of finding the polar equation of the curve amounts to solving the differential equation

$$\frac{1}{p^2} = \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + u^2 = P(u). \tag{8.4}$$

Equation (8.4) arises in many other contexts. Here are two examples.

- If a particle moves in a spherically symmetric potential $V(\frac{1}{r})$ per unit mass and has conserved energy per unit mass $\mathcal{E}$ and conserved angular momentum per unit mass h then its orbit satisfies (8.4) with

$$P(u) = \frac{2}{h^2}\left(\mathcal{E} - V\left(\frac{1}{u}\right)\right). \tag{8.5}$$

- The scale factor $a(t)$ of Friedmann–Lemaitre universe with energy density $\rho(a)$ (including any contribution due to a cosmological term) satisfies

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = 4\pi G\rho(a) + \frac{(1-k)}{a^2}, \tag{8.6}$$

which may be brought to the form (8.4) with $u = \frac{1}{a}$, $\phi = \mathrm{d}\eta = a\mathrm{d}t$, and

$$P(u) = 4\pi G a^2 \rho(a) + (1-k)a^2. \tag{8.7}$$

8.1 Affine and rescaling of angle

Suppose $u_1 = f(\phi)$ satisfies

$$\left(\frac{\mathrm{d}u_1}{\mathrm{d}\phi}\right)^2 + u_1^2 = P_1(u_1). \tag{8.8}$$

Let $u_2 = Af(\lambda\phi) + B$. Then

$$\left(\frac{\mathrm{d}u_2}{\mathrm{d}\phi}\right)^2 + u_2^2 = P_2(u_2), \tag{8.9}$$

with

$$P_2(u_2) = A^2\lambda^2 P_1\left(\frac{u_2}{A} - B\right) + 2\lambda^2 ABu_2 + (1-\lambda^2)u_2^2 - \lambda^2 A^2 B^2. \tag{8.10}$$

We may compose two such tranformations to get a third:

$$A_3 = A_2 A_1, \qquad \lambda_3 = \lambda_2\lambda_1, \qquad B_3 = B_2 + A_s B_1. \tag{8.11}$$

Thus the set of central potentials is acted upon by a three-parameter group. The first two formulas are commutative but the second is not. In the above the second transformation A_2, B_2, λ_2 follows the first A_1, B_1, λ_1.

A matrix representation may be given

$$\begin{pmatrix} u \\ \phi \\ 1 \end{pmatrix} \to \begin{pmatrix} A & 0 & B \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \phi \\ 1 \end{pmatrix} \tag{8.12}$$

which reveals the three-dimensional group as $\mathbb{R}^\times$ corresponding to λ times the semi-direct product $\mathbb{R}^\times \ltimes \mathbb{R}^+$ corresponding to A and B. Acting on $P(u)$, we have

$$P(u) \to \lambda^2 A^2 \left(P\left(\frac{u}{A} + B\right) - \left(\frac{u}{A} + B\right)^2\right) + u^2. \tag{8.13}$$

Or if $Q(u) = P(u) - u^2$, then

$$Q(u) \to \lambda^2 A^2 Q\left(\frac{u}{A} + B\right), \tag{8.14}$$

which says that $Q(u)$ transforms under pull-back composed with mutiplication by $A^2\lambda^2$.

8.2 Non-linear transformations

The simplest of these is to raise u to a power: If

$$u_2 = (u_1)^n, \tag{8.15}$$

then

$$P_2(u) = n^2 u^{2\frac{n-1}{n}} P_1\left(u^{\frac{1}{n}}\right) + (1-n^2)u^2. \tag{8.16}$$

The case $n = 1$ is particularly simple:

$$P_2(u) = u^4 P_1\left(\frac{1}{u}\right). \tag{8.17}$$

8.3 Chebyshev's theorem

From (8.4) we deduce that

$$\phi = \int \frac{\mathrm{d}u}{\sqrt{P(u) - u^2}} \,. \tag{8.18}$$

Chebyshev's theorem states that for rational numbers p, q, r $(r \neq 0)$ [1] and nonzero real numbers α, β, the integral

$$I = \int x^p \left(\alpha + \beta x^r\right)^q \,\mathrm{d}x = \int \left(\alpha x^{\frac{p}{q}} + \beta x^{\frac{p}{q}+r}\right)^q \,\mathrm{d}x \tag{8.19}$$

is elementary if and only if at least one of the quantities

$$\frac{p+1}{r}, \quad q, \quad \frac{p+1}{r} + q, \tag{8.20}$$

is an integer. In our case $q = -\frac{1}{2}$ and so integrability in finite terms is possible if

$$P(u) = u^2 + \alpha u^{\frac{p}{q}} + \beta u^{\frac{p}{q}+r}\,, \tag{8.21}$$

where $\frac{p+1}{r}$ is an integer or half integer.

8.4 Examples

8.4.1 Circles

With respect to a point distance A from a circle of radius R

$$r^2 = A^2 - R^2 + 2Rp, \qquad \frac{1}{p^2} = \frac{4R^2u^4}{\left(1 + (R^2 - A^2)u^2\right)^2}. \tag{8.22}$$

8.4.2 Conics

An ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1, \tag{8.23}$$

with respect to the focus

$$\frac{1}{p^2} = \frac{1}{B^2}\left(\frac{1}{2Au} - 1\right), \tag{8.24}$$

and with respect to the centre

$$\frac{1}{p^2} = \frac{A^2 + B^2}{A^2B^2} - \frac{1}{u^2}. \tag{8.25}$$

A hyperbola

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1, \tag{8.26}$$

with respect to the focus

$$\frac{1}{p^2} = \frac{1}{B^2}\left(1 - \frac{1}{2Au}\right), \tag{8.27}$$

[1] p in this section should not be confused for the perpendicular distance from the origin to the tangent.

and with respect to the centre

$$\frac{1}{p^2} = \frac{A^2 - B^2}{A^2 B^2} + \frac{1}{u^2}. \tag{8.28}$$

A parabola

$$y^2 = 4Ax, \tag{8.29}$$

with respect to the focus

$$\frac{1}{p^2} = \frac{u}{A}, \tag{8.30}$$

with rsepect to its vertex

$$A^2(r^2 - p^2)^2 = p^2(r^2 + 4A^2)(p^2 + 4A^2). \tag{8.31}$$

8.4.3 Algebraic spirals

$$u = \phi^{-m}, \quad \frac{1}{p^2} = u^2 + \frac{1}{m^2}u^{2m+2}. \tag{8.32}$$

- $m = 1$: Archimedes Spiral.
- $m = 1$: Hyperbolic or Reciprocal Spiral.
- $m = 2$: Galilei's Spiral.
- $m = -\frac{1}{2}$: Cotes's Lituus.
- $m = \frac{1}{2}$: Fermat's Spiral.

8.4.4 Logarithmic or equiangular spiral

- $u = \mathrm{e}^{-\phi \cot \alpha} \Longleftrightarrow p = r \sin \alpha$, where α is the angle beween the curve and the radius vector.

8.4.5 Sinusoidal spirals

$$u = (\cos k\phi)^m, \quad \frac{1}{p^2} = k^2 m^2 u^{2(1-\frac{1}{m})} + (1 - k^2 m^2) u^2. \tag{8.33}$$

If $m = -1$ we obtain the Rhodenea $u = \frac{1}{\cos k\phi}$.

$k = 3$ gives the Trifolium.

$k = 2$ the Quadrifolium and

$k = \frac{1}{3}$ is the pedal of the Cardioid.

If $m = 1$ we obtain the Epi-spirals $u = \cos k\phi$.

If $m = -\frac{1}{k}$ we have $u = (\cos k\phi)^{-\frac{1}{k}}$ and $\frac{1}{p^2} = u^{2(1+k)}$.

Some particlular sinusoidal spirals are:

- $m = 1, k = \frac{1}{2}$. The Trisectrix of De Longe $u = \cos \frac{\phi}{2}$, $\frac{1}{p^2} = \frac{1}{4} + \frac{3}{4}u^2$.
- $m = 1, k = \frac{1}{3}$. Maclaurin's Trisectrix $u = \cos \frac{\phi}{3}$, $\frac{1}{p^2} = \frac{1}{9} + \frac{8}{9}u^2$.

- $m = 3, k = 1$. The Cubic Duplicatrix $u = \cos^3\phi$, $\frac{1}{p^2} = 9u^{\frac{4}{3}} - 8u^2$.
- $m = 2, k = 1$. The Kampyle $u = \cos^2\phi$, $\frac{1}{p^2} = 4u - 3u^2$.
- $m = \frac{3}{2}, k = 1$. The Witch of Agnesi $u = \cos^{\frac{3}{2}}\phi$, $\frac{1}{p^2} = 4u^{\frac{2}{3}} - \frac{5}{4}u^2$.
- $m = 2, k = 1$. The Cruciform $u = \cos 2\phi$, $\frac{1}{p^2} = 4 - 3u^2$.
- $m = -2, k = 1$. The Oeuf Double $u = \cos^{-2}\phi$, $\frac{1}{p^2} = 4u^3 - 3u^2$.
- $m = -3, k = \frac{1}{3}$. Cayley's Sextic $u = \frac{1}{\cos^3\frac{\phi}{3}}$, $\frac{1}{p^2} = u^{\frac{8}{3}}$.

8.4.6 Black holes

As central orbits, the first two examples below are related by Arnold–Bohlin duality [39, 40] while the third is self-dual. They arise in the theory of null geodesics of spherically symmetric Ricci flat black hole metrics [41].

If $u = \frac{\cosh\phi+2}{\cosh\phi-1}$ or $u = \frac{\cosh\phi-2}{\cosh\phi+1}$ we have $\frac{1}{p^2} = \frac{1}{3} + \frac{2}{3}u^3$. These are particular null geodesics of a four-dimensional black hole.

If $u^2 = \frac{\cosh 2\phi+1}{\cosh 2\phi-2}$ or $u = \frac{\cosh 2\phi-1}{\cosh\phi+2}$ we have $\frac{1}{p^2} = \frac{2}{3} + \frac{1}{3}u^6$. These are particular null geodesics of a seven-dimensional black hole.

If $u = \tanh\left(\frac{\phi}{\sqrt{2}}\right)$ or $u = \coth\left(\frac{\phi}{\sqrt{2}}\right)$ we have $\frac{1}{p^2} = 2(1+u^4)$. These are particular null geodesics of a five-dimensional black hole.

8.4.7 Other examples

Freeth's Nephroid is

$$r = \left(1 + 2\sin\frac{\phi}{2}\right), \tag{8.34}$$

whence

$$\frac{1}{p^2} = \frac{u^2}{4}(7 - 2u + u^2). \tag{8.35}$$

8.4.8 Some hyperbolic spirals

$$u = (\tanh k\phi)^m, u = (\coth k\phi)^m, \frac{1}{p^2} = m^2k^2\left(u^{2\frac{m-1}{m}} + u^{2\frac{m+1}{m}}\right) + (1 - 2m^2k^2)u^2. \tag{8.36}$$

$$u = (\cosh k\phi)^m, \quad \frac{1}{p^2} = -m^2k^2u^{2\frac{m-1}{m}} + (1 + m^2k^2)u^2, \tag{8.37}$$

$$u = (\sinh k\phi)^m, \quad \frac{1}{p^2} = m^2k^2u^{2\frac{m-1}{m}} + (1 + m^2k^2)u^2. \tag{8.38}$$

If $m = 1$ we get the two types of Poinsot Spirals for which

$$\frac{1}{p^2} = (k^2 + 1)^2 \mp k^2, \tag{8.39}$$

respectivley.

8.4.9 Further examples

- The astroid is

$$\frac{1}{p^2} = \frac{3u^2}{u^2-1}. \tag{8.40}$$

- The nehproid is

$$\frac{1}{p^2} = \frac{3u^2}{4-16u^2}. \tag{8.41}$$

- The deltoid is

$$\frac{1}{p^2} = \frac{8u^2}{9u^2-1}. \tag{8.42}$$

$$u = (\tan k\phi)^m\,, \quad \frac{1}{p^2} = m^2k^2\left(u^{2(1+\frac{1}{m})} + u^{2(1-\frac{1}{m})}\right) + (2m^2k^2+1)u^2. \tag{8.43}$$

- Kappa $u = \cot\phi$, $m = -1, k = 1$, $\frac{1}{p^2} = 1 + 3u^2 + u^4$.

- Moulin á vent $u = \cot 2\phi$, $m = 1, k = 2$, $\frac{1}{p^2} = 4 + 9u^2 + 4u^4$.